

# HOPF ALGEBRAS OF TYPES $U_q(sl_n)'$ AND $O_q(SL_n)'$ WHICH GIVE RISE TO CERTAIN INVARIANTS OF KNOTS, LINKS AND 3-MANIFOLDS

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**ABSTRACT.** In this paper we determine when Lusztig's  $U_q(sl_n)'$  has all the desired properties necessary to define invariants of knots, links and 3-manifolds. Specifically, we determine when it is ribbon, unimodular and factorizable. We also compute the integrals and distinguished elements involved.

Finite dimensional Hopf algebras arise in a variety of settings in the theory of quantum groups, and in this context have interesting connections with representation theory, topology and physics. Perhaps one of the most intriguing phenomena is the relationship between finite dimensional quasitriangular Hopf algebras and invariants of knots, links and 3-manifolds, as reflected in a series of papers by Hennings, Kauffman, Radford, Reshetikhin and Turaev [H, K, KR1, KR2, R1, R2, RT]. Reshetikhin and Turaev have defined a special quantum group termed "Ribbon". This is a quasitriangular Hopf algebra which contains a special element,  $v$ , that reflects the crucial axioms for the construction of invariants of framed links, and hence of 3-manifolds. In fact Reshetikhin, Turaev and Kauffman have rediscovered the celebrated Jones polynomial using such a quantum group [K, RT].

Kauffman has constructed a regular invariant of unoriented knots and links [K, KR2]. The necessary ingredients for this invariant are a so-called quantum algebra  $A$ , and a trace-like function  $\chi$  from  $A$  to the ground field. Radford has studied these knot invariants which arise from finite dimensional ribbon Hopf algebras  $(A, R, v)$  [R1]. A major role here is played by the important structure elements of finite dimensional Hopf algebras, namely the right and left integrals. When these elements coincide the Hopf algebra is called unimodular. Radford has proved that if  $(A, R, v)$  is ribbon and unimodular over a field  $k$ , then it is a quantum algebra, and there is a trace-like functional  $\chi$  which is also a (generalized) right integral of  $A^*$ . This implies in turn that any representation of a finite dimensional ribbon Hopf algebra gives rise to a quantum algebra—hence the importance of these Hopf algebras as sources for various invariants. Furthermore, under very general conditions, for example when the ribbon Hopf algebra  $(A, R, v)$  is factorizable,  $\chi$  can also be used to construct invariants of 3-manifolds. Hennings was the first to do so [H]. He has proved that it can be done if and only if  $\lambda(v)\lambda(v^{-1}) \neq 0$ , where  $\lambda$  is a non-zero right integral for  $A^*$  [H, K, R1].

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One of the most important families of quantum groups is the family of quantized enveloping algebras,  $U_q(\mathcal{G})$ . The first quantum group in this family was a deformation of the enveloping algebra  $U(sl_n)$ , denoted by  $U_q(sl_n)$ . A generalization of  $U_q(sl_n)$  is  $U_q(\mathcal{G})$ , where  $\mathcal{G}$  is any complex simple Lie algebra. When  $q$  is *not* a root of unity  $U_q(\mathcal{G})$  is quasitriangular and ribbon, and its representation theory is remarkably similar to that of  $U(\mathcal{G})$ . However, when  $q$  is a root of unity the situation changes dramatically.  $U_q(\mathcal{G})$  is no longer quasitriangular and ribbon, and the representation theory of  $U(\mathcal{G})$  is no longer effective. When  $q$  is a root of unity the quantum group is a more complicated object, and these complications are similar to those encountered in the representation theory of semisimple algebraic groups in positive characteristic (even though the quantum group is in characteristic zero). In this case one can consider the quotient Hopf algebras  $U_q(\mathcal{G})'$ , introduced by Lusztig [L], which are finite dimensional non-semisimple Hopf algebras with a representation theory similar to that of  $U(\mathcal{G})$ . It is well known that  $U_q(sl_2)'$  is quasitriangular and ribbon for any  $q$ . This quantum group has two other properties important for the construction of invariants of links and 3-manifolds: it is unimodular and factorizable. Therefore, a natural question is: when does  $U_q(sl_n)'$  have all the desired properties necessary to define knot, link and 3-manifold invariants? The first step in this direction has been taken in [GW], where we proved:

**Theorem 3.7** ([GW]). *A necessary and sufficient condition for  $U_q(sl_n)'$  to be quasitriangular is:*

$$(n, N) = 1 \text{ or } 2,$$

where  $N$  is the order of  $q^{1/2}$ .

Our purpose in this paper is to proceed and determine when  $U_q(sl_n)'$  is ribbon, unimodular and factorizable. In the process of analyzing the factorizability of  $U_q(sl_n)'$  we prove a new useful criterion for the factorizability of a quantum group. It involves the integrals of the quantum group and its dual Hopf algebra.

We prove:

**Theorem 2.1.5.** *The element*

$$\Lambda := \left( \sum_{g \in G(U)} g \right) \prod_{(i,j)} E_{ij}^{r-1} \prod_{(i,j)} F_{ij}^{r-1}$$

*is a non-zero two-sided integral of  $U_q(sl_n)'$ . Thus  $U_q(sl_n)'$  is unimodular for all  $n$  and  $q$ .*

We determine when  $U_q(sl_n)'$  is ribbon in:

**Theorem 2.2.6.**  *$U_q(sl_n)'$  is a ribbon Hopf algebra if and only if  $(n, N) = 1$  or  $2$ ; that is,  $U_q(sl_n)'$  is ribbon if and only if it is quasitriangular.*

Thus, we have determined when  $U_q(sl_n)'$  can be used to compute invariants of knots and links.

We next focus on the factorizability of  $U_q(sl_n)'$ . We start by proving a general result:

**Theorem 2.3.2.** *Let  $(A, R_f)$  be a finite dimensional quasitriangular Hopf algebra, and set  $F := f^* * f$ . Let  $\Lambda, \lambda$  be non-zero right integrals of  $A$  and  $A^*$  respectively. Then  $A$  is factorizable if and only if  $F(\lambda) = \alpha\Lambda$  for some non-zero scalar  $\alpha \in k$ .*

Thus we are able to prove:

**Theorem 2.3.4.** *Let  $N$  be the order of  $q^{1/2}$ . Then  $U_q(sl_n)'$  is factorizable if and only if  $(n, N) = 1$  and  $N$  is odd; that is, if and only if it is minimal quasitriangular.*

Thus, we have determined when  $U_q(sl_n)'$  can be used to compute invariants of 3-manifolds.

In order to prove Theorems 2.2.6 and 2.3.2 we need to compute a non-zero right integral of  $U_q(sl_n)' = O_q(SL_n)'$  and the distinguished grouplike element of  $U_q(sl_n)'$ . Furthermore, these two elements will give rise to the trace-like function of H-K-R invariants. We prove:

**Theorem 2.2.3.** *Set  $N := \{0, 1, \dots, N-1\}^{n-1}$ . Then the element*

$$\lambda := \left( \prod_{(i,j) \in J} x_{ij}^{r-1} \right) \left( \sum_{\mathbf{m} \in \mathbf{N}} \prod_{i=1}^n x_{ii}^{m_i} \right)$$

*is a non-zero right integral of  $O_q(SL_n)'$ .*

**Theorem 2.2.5.** *The element*

$$g := K_1^{4(n-1)} \left( \prod_{i=2}^{n-2} K_i^{8(n-2)} \right) K_{n-1}^{4(n-1)}$$

*is the distinguished grouplike element of  $U_q(sl_n)'$ .*

## 1. PRELIMINARIES

Let  $A$  be a finite dimensional Hopf algebra over a field  $k$  with antipode  $s$ . The transpose actions on  $A^*$ , described by

$$(a \cdot p)(b) = p(ba) \quad \text{and} \quad (p \cdot a)(b) = p(ab)$$

for  $a, b \in A$  and  $p \in A^*$ , give  $A^*$  an  $A$ -bimodule structure. Similarly  $A$  is an  $A^*$ -bimodule, where

$$p \rightharpoonup a = \sum a_{(1)} p(a_{(2)}) \quad \text{and} \quad a \leftharpoonup p = \sum p(a_{(1)}) a_{(2)}$$

for  $p \in A^*$  and  $a \in A$ , where we write  $\Delta(a) = \sum a_{(1)} \otimes a_{(2)}$ .

“Twisting” multiplication and comultiplication in  $A$  gives rise to Hopf algebras  $A^{op}$  and  $A^{cop}$ , respectively. As a coalgebra  $A^{op} = A$ , and multiplication in  $A^{op}$  is defined by  $a \cdot b = ba$  for  $a, b \in A$ . As an algebra  $A^{cop} = A$ , and comultiplication in  $A^{cop}$  is defined by  $\Delta^{cop}(a) = \sum a_{(2)} \otimes a_{(1)}$  for  $a \in A$ . The antipode  $s$  is an algebra and a coalgebra anti-isomorphism. Thus  $A^{op\,cop}$  is a Hopf algebra with antipode  $s$ , and  $A^{op}$ ,  $A^{cop}$  are Hopf algebras with antipode  $s^{-1}$ . Thus  $A \cong A^{op\,cop}$  and  $A^{op} \cong A^{cop}$  as Hopf algebras.

A non-zero element  $g$  in  $A$  is said to be a *grouplike element* if  $\Delta(g) = g \otimes g$ . The group of grouplike elements of  $A$  is denoted by  $G(A)$ . Since  $A$  is finite dimensional,  $G(A^*) = \text{Alg}_k(A, k)$ .

Let  $\Lambda \in A$  be a non-zero left integral for  $A$ , and let  $\lambda \in A^*$  be a non-zero right integral for  $A^*$ . The left integrals for  $A$  form a one dimensional ideal of  $A$ . Hence there is a unique  $\alpha \in G(A^*)$  such that  $\Lambda a = \alpha(a)\Lambda$  for all  $a \in A$ . Likewise there is a unique  $g \in G(A^{**}) = G(A)$  such that  $p\lambda = p(g)\lambda$  for all  $p \in A^*$ . We call  $g$  the *distinguished grouplike element* of  $A$  and  $\alpha$  the *distinguished grouplike element* of  $A^*$ . These grouplike elements play a fundamental role in the structure of  $A$ . We say that  $A$  is *unimodular* if the ideal of left integrals for  $A$  equals the ideal of right integrals for  $A$ . Thus  $A$  is unimodular if and only if  $\alpha = \varepsilon$ , and  $A^*$  is unimodular if and only if  $g = 1$ .

Let  $R = \sum R^{(1)} \otimes R^{(2)} \in A \otimes A$ , and define a linear map

$$f_R : A^* \rightarrow A, \quad f_R(p) = \sum p(R^{(1)})R^{(2)},$$

for  $p \in A^*$ . Then the pair  $(A, R)$  is said to be a *quasitriangular* Hopf algebra if the following axioms hold ( $r = R$ ):

- (QT.1):  $\sum \Delta(R^{(1)}) \otimes R^{(2)} = \sum R^{(1)} \otimes r^{(1)} \otimes R^{(2)}r^{(2)}$ ,
- (QT.2):  $\sum \varepsilon(R^{(1)})R^{(2)} = 1$ ,
- (QT.3):  $\sum R^{(1)} \otimes \Delta^{cop}(R^{(2)}) = \sum R^{(1)}r^{(1)} \otimes R^{(2)} \otimes r^{(2)}$ ,
- (QT.4):  $\sum R^{(1)}\varepsilon(R^{(2)}) = 1$  and
- (QT.5):  $(\Delta^{cop}(a))R = R(\Delta(a))$  for all  $a \in A$ ;

or equivalently if  $f_R : A^* \rightarrow A^{cop}$  is a Hopf algebra map and (QT.5) is satisfied. Observe that (QT.5) is equivalent to

$$(\text{QT.5})': \sum p_{(1)}(a_{(2)})a_{(1)}f_R(p_{(2)}) = \sum p_{(2)}(a_{(1)})f_R(p_{(1)})a_{(2)} \text{ for all } p \in A^* \text{ and } a \in A.$$

Note that the map  $f_R^* : A^{*op} \rightarrow A$  is a Hopf algebra map which satisfies

$$f_R^*(p) = \sum p(R^{(2)})R^{(1)}$$

for all  $p \in A^*$  and  $a \in A$ .

Conversely, let  $f : A^{*cop} \rightarrow A$  be a Hopf algebra map and let  $R_f \in A \otimes A$  be the corresponding element via the canonical vector spaces isomorphism between  $\text{Hom}_k(A^*, A)$  and  $A \otimes A$  (i.e.  $f = f_{R_f}$ ). We say that  $f$  determines a quasitriangular structure on  $A$  if  $(A, R_f)$  is quasitriangular, or equivalently, if  $f$  satisfies (QT.5)'.

Let  $(A, R)$  be quasitriangular, and write  $R = \sum R^{(1)} \otimes R^{(2)}$  in the shortest possible way. Set  $B := \text{sp}_k\{R^{(1)}\}$  and  $H := \text{sp}_k\{R^{(2)}\}$ . Note that  $B = \text{Im}(f_R^*)$  and  $H = \text{Im}(f_R)$ ; hence  $B$  and  $H$  are sub Hopf algebras of  $A$ . Let  $A_R$  be the sub Hopf algebra of  $A$  generated by  $B$  and  $H$ . Then  $(A_R, R)$  is a quasitriangular Hopf algebra. If  $A = A_R$ , then  $(A, R)$  is called a *minimal* quasitriangular Hopf algebra. We shall also say that  $A$  is a minimal quasitriangular Hopf algebra if there exists  $R \in A \otimes A$  such that  $(A, R)$  is a minimal quasitriangular Hopf algebra.

Let  $(A, R)$  be a finite dimensional quasitriangular Hopf algebra with antipode  $s$  over  $k$ . Set

$$u := \sum s(R^{(2)})R^{(1)}.$$

Since  $R$  is invertible, it follows that  $u$  is invertible as well. By [D1],

$$\begin{aligned} \Delta(u) &= (u \otimes u)(R^\tau R)^{-1}, \quad \varepsilon(u) = 1, \\ s^2(a) &= uau^{-1} \quad \text{for all } a \in A. \end{aligned}$$

A finite dimensional ribbon Hopf algebra over  $k$  is a triple  $(A, R, v)$  where  $(A, R)$  is a finite dimensional quasitriangular Hopf algebra over  $k$  and  $v \in A$  satisfies the following:

- (R.0):  $v$  is in the center of  $A$ ,
- (R.1):  $v^2 = us(u)$ ,
- (R.2):  $s(v) = v$ ,
- (R.3):  $\varepsilon(v) = 1$  and
- (R.4):  $\Delta(v) = (v \otimes v)(R^\tau R)^{-1} = (R^\tau R)^{-1}(v \otimes v)$ .

Observe that  $G := uv^{-1}$  is a grouplike element of  $A$ . It is called *the special grouplike element* of  $A$ . Ribbon Hopf algebras were introduced and studied by Reshetikhin and Turaev [RT].

A finite dimensional quasitriangular Hopf algebra  $(A, R)$  over  $k$  is said to be *factorizable* if the (linear) map  $(f_R)^* * f_R : A^* \rightarrow A$  is a linear isomorphism (where  $*$  stands for convolution). A factorizable Hopf algebra is always minimal and unimodular [R1]. If moreover  $(A, R, v)$  is a factorizable ribbon Hopf algebra and  $\lambda$  is a nonzero right integral of  $A^*$ , then  $\lambda(v)\lambda(v^{-1}) \neq 0$  [H, KR2, R1].

Hennings-Kauffman-Radford (H-K-R) invariants arising from finite dimensional unimodular ribbon Hopf algebras are defined as follows [K, KR2, R1]: Let  $(A, R, v)$  be a finite dimensional unimodular ribbon Hopf algebra and let  $\lambda$  be a non-zero right integral of  $A^*$ . Any knot  $K$  gives rise to an element  $G^d w$  in  $A$  according to a diagrammatic representation of it in the plane, where  $G = v^{-1}u$  is the special grouplike element of  $A$ . Then,  $Tr(K) := (\lambda \cdot G)(G^d w)$  is a regular isotopy invariant of knots [K, KR2]. If moreover  $\lambda(v)\lambda(v^{-1}) \neq 0$  (e.g when  $(A, R, v)$  is factorizable), then  $Tr$  can be used to define a 3-manifolds invariant [H, K, KR2]. More precisely, for a knot  $K$  let

$$Inv(K) := [\lambda(v)\lambda(v^{-1})]^{-1/2} [\lambda(v)/\lambda(v^{-1})]^{-\sigma(K)/2} Tr(K),$$

where  $\sigma(K)$  denotes the signature of the matrix of linking numbers of the components of  $K$  (with framing numbers on the diagonal); then  $Inv(K)$  is an invariant of the 3-manifold obtained by doing framed surgery on  $K$  in the blackboard framing.

In the literature there are several versions of  $U_q(sl_n)$  [D2, J]. We shall use the following one. Let  $A = (a_{ij})$  be the Cartan matrix of  $sl_n$ , that is,  $a_{ij} = -1$  if  $|i - j| = 1$ ,  $a_{ii} = 2$  and  $a_{ij} = 0$  otherwise. Let  $q \in k^*$  be such that  $q^4 \neq 1$ . As an algebra,  $U_q(sl_n)$  is generated by  $K_i$ ,  $E_i$  and  $F_i$  for  $1 \leq i \leq n - 1$  subject to the following relations:

- (U.1):  $K_i K_j = K_j K_i$  for all  $1 \leq i, j \leq n - 1$ ,
- (U.2):  $K_i E_j = q^{\frac{a_{ij}}{2}} E_j K_i$  and  $K_i F_j = q^{-\frac{a_{ij}}{2}} F_j K_i$ ,
- (U.3):  $[E_i, F_j] = \delta_{ij} \frac{K_i^2 - K_i^{-2}}{q - q^{-1}}$ ,
- (U.4):  $E_i^2 E_j - (q + q^{-1}) E_i E_j E_i + E_j E_i^2 = 0$  and
- (U.5):  $F_i^2 F_j - (q + q^{-1}) F_i F_j F_i + F_j F_i^2 = 0$ .

The coalgebra structure is determined by

$$\begin{aligned} \Delta(K_i) &= K_i \otimes K_i, & \Delta(E_i) &= E_i \otimes K_i + K_i^{-1} \otimes E_i, \\ \Delta(F_i) &= F_i \otimes K_i + K_i^{-1} \otimes F_i, \\ \varepsilon(K_i) &= 1 \quad \text{and} \quad \varepsilon(E_i) = \varepsilon(F_i) = 0, \end{aligned}$$

and the antipode  $S$  is determined by

$$S(K_i) = K_i^{-1}, \quad S(E_i) = -q E_i \quad \text{and} \quad S(F_i) = -q^{-1} F_i.$$

Let  $\omega := q^{1/2}$ . Following Jimbo [J], set

$$\begin{aligned} E_{i,i+1} &:= E_i, & F_{i,i+1} &:= F_i, \\ E_{i,j} &:= \omega^{-1} E_{i,j-1} E_{j-1,j} - \omega E_{j-1,j} E_{i,j-1}, & j - i > 1, \\ F_{i,j} &:= \omega^{-1} F_{i,j-1} F_{j-1,j} - \omega F_{j-1,j} F_{i,j-1}, & j - i > 1. \end{aligned}$$

Let  $K_{i,j} := K_i \cdots K_{j-1}$  for  $i < j$ . Observe that

$$(1) \quad \begin{aligned} E_{i,j} K_r &= q^{\frac{1}{2}(\delta_{i,r+1} - \delta_{i,r} + \delta_{j,r} - \delta_{j,r+1})} K_r E_{i,j}, \\ F_{i,j} K_r &= q^{-\frac{1}{2}(\delta_{i,r+1} - \delta_{i,r} + \delta_{j,r} - \delta_{j,r+1})} K_r F_{i,j}. \end{aligned}$$

Compare with [Y, page 510 (4)]. Suppose now that  $q$  is a root of unity, and let  $N$  be the order of  $\omega$  and  $r$  be the order of  $q^2$ . Following Lusztig [L], we define  $U_q(sl_n)'$  to be the finite dimensional quotient of  $U_q(sl_n)$  over the Hopf ideal  $I$ , generated by  $K_i^N - 1$ ,  $E_{i,j}^r$  and  $F_{i,j}^r$ . Combining results of [L, Y] yields the following linear basis of  $U_q(sl_n)'$ :

**Theorem 1.1.** *Let  $q \in k^*$  be such that  $q^4 \neq 1$ . Then the elements*

$$\prod_{i=1}^{n-1} K_i^{l_i} \prod_{i,j} E_{i,j}^{m(i,j)} \prod_{i,j} F_{i,j}^{k(i,j)},$$

where  $0 \leq m(i,j)$ ,  $k(i,j) \leq r-1$  and  $0 \leq l_i \leq N-1$ , form a linear basis of  $U_q(sl_n)'$ .

Recall from [T, Section 5.4] the definition of  $O_q(SL_n)'$ . As an algebra it is generated by  $x_{ij}$ , where  $1 \leq i, j \leq n$ , subject to the following relations:

- (O.1):  $x_{ii}^N = 1$  for all  $1 \leq i \leq n$ ,
- (O.2):  $x_{ij}^r = 0$  if  $i \neq j$ ,
- (O.3):  $x_{ik}x_{ij} = qx_{ij}x_{ik}$  if  $j < k$ ,
- (O.4):  $x_{jk}x_{ik} = qx_{ik}x_{jk}$  if  $i < j$ ,
- (O.5):  $x_{jk}x_{il} = x_{il}x_{jk}$  if  $i < j$  and  $k < l$ ,
- (O.6):  $x_{jl}x_{ik} - x_{ik}x_{jl} = (q - q^{-1})x_{il}x_{jk}$  if  $i < j$  and  $k < l$ , and
- (O.7):  $\det_q(X) = \sum_{\sigma} (-q)^{-l(\sigma)} x_{1,\sigma(1)} \cdots x_{n,\sigma(n)} = 1$ .

The last sum is over all permutations  $\sigma$  of  $n$  letters, where  $l(\sigma)$  denotes the number of inversions and  $X = (x_{ij})$ . The coalgebra structure is determined by

$$\Delta(x_{ij}) = \sum_{s=1}^n x_{is} \otimes x_{sj} \quad \text{and} \quad \varepsilon(x_{ij}) = \delta_{ij}.$$

Let  $Y$  denote the  $(n-1) \times (n-1)$  generic matrix obtained by deleting the  $i^{th}$  row and  $j^{th}$  column of  $X$ , and set  $A_{i,j} := \det_q(Y)$ . The antipode  $S$  of  $O_q(SL_n)'$  is determined by

$$S(x_{ij}) = (-q)^{j-i} A_{j,i}.$$

In [T, Section 5.4] it is proved that

$$\left\{ \prod_{i,j} x_{ij}^{m(i,j)} \mid 0 \leq m(i,i) \leq N-1, \quad 0 \leq m(i,j) \leq r-1 \quad \text{if } i \neq j \right\}$$

is a linear basis of  $O_q(SL_n)'$ , where the product is taken relative to an arbitrary fixed total ordering.

The pairing between  $O_q(SL_n)'$  and  $U_q(sl_n)'$  is given by

$$(2) \quad \begin{aligned} \langle x_{ij}, K_s \rangle &= \delta_{i,j} \omega^{\delta_{s,i} - \delta_{s+1,i}}, & \langle x_{ij}, E_s \rangle &= \delta_{i,s} \delta_{i+1,j}, \\ \langle x_{ij}, F_s \rangle &= \delta_{j,s} \delta_{j+1,i}, \end{aligned}$$

where  $1 \leq s \leq n-1$ .

In the following we collect a few results from [GW].

**Theorem 1.2** ([GW, Theorems 3.7, 4.1]). 1.  $U_q(sl_n)'$  is quasitriangular if and only if  $(n, N) = 1$  or 2.  
 2.  $U_q(sl_n)'$  is minimal quasitriangular if and only if  $(n, N) = 1$  and  $N$  is odd. In this case it admits exactly two non-isomorphic minimal quasitriangular structures. One structure  $(U_q(sl_n)', R_f)$  is determined as follows. Set  $g_{ij} := \prod_{t=1}^{n-1} K_t^{2(n-t)/n} \prod_{t=1}^{i-1} K_t^{-2} \prod_{t=i}^{j-1} K_t^{-1}$  for all  $1 \leq i \leq j \leq n$ . Then,

$$\begin{aligned} f(x_{ii}) &= g_{ii}, & 1 \leq i \leq n, \\ f(x_{ij}) &= 0, & i > j, \\ f(x_{ij}) &= (\omega - \omega^{-3})^{j-i} (-\hat{q})^{i-j+1} g_{ij} F_{ij}, & i < j. \end{aligned}$$

Its dual map  $f^*$  is given by

$$\begin{aligned} f^*(x_{ii}) &= g_{ii}, & 1 \leq i \leq n, \\ f^*(x_{ij}) &= 0, & i < j, \\ f^*(x_{ij}) &= (\omega^{-1} - \omega^3)^{j-i} (-\hat{q})^{i-j+1} g_{ji} E_{ji}, & i < j. \end{aligned}$$

The other structure is determined by exchanging  $E$  and  $F$ .

## 2. $U_q(sl_n)'$ , $O_q(SL_n)'$ AND H-K-R INVARIANTS

In this section we first describe a non-zero two-sided integral for  $U_q(sl_n)'$ , and thus prove that  $U_q(sl_n)'$  is unimodular for all  $n$  and  $q$ . Second, we describe a non-zero right integral and a non-zero left integral for  $O_q(SL_n)'$  and compute the distinguished grouplike element of  $U_q(sl_n)'$  for all  $n$  and  $q$ . Third, we prove that  $U_q(sl_n)'$  is ribbon whenever it is quasitriangular. Thus,  $U_q(sl_n)'$  can be used to compute H-K-R invariants of knots and links. We conclude by finding a necessary and sufficient condition for  $U_q(sl_n)'$  to be factorizable. Thus, we prove that when this condition is realized,  $U_q(sl_n)'$  can be used to compute H-K-R invariants of 3-manifolds.

**2.1.  $U_q(sl_n)'$  is unimodular.** In what follows we fix  $n$  and  $q$  and set  $U := U_q(sl_n)'$ ,  $U^+ := k\langle K_i, E_{ij} \rangle$  and  $U^- := k\langle K_i, F_{ij} \rangle$ . We also set  $T := \{(m, l) | 1 \leq m < l \leq n-1\}$  and order it lexicographically, reading left to right.

**Lemma 2.1.1.** Let  $(m, l) \in T$ . Then for all  $(s, t) < (m, l)$  the following holds:

$$E_{ml} \left( \prod_{(i,j) < (m,l)} E_{ij}^{r-1} \right) = \alpha \left( \prod_{(i,j) \leq (s,t)} E_{ij}^{r-1} \right) E_{ml} \left( \prod_{(s,t) < (i,j) < (m,l)} E_{ij}^{r-1} \right)$$

and

$$\left( \prod_{(m,l) < (i,j)} E_{ij}^{r-1} \right) E_{ml} = \beta \left( \prod_{(m,l) < (i,j) < (s,t)} E_{ij}^{r-1} \right) E_{ml} \left( \prod_{(s,t) \leq (i,j)} E_{ij}^{r-1} \right)$$

for some scalars  $\alpha, \beta \in k$ .

*Proof.* Set  $\hat{q} := q - q^{-1}$ . Recall from [Y, page 509] that

$$\begin{aligned} (Y.1) \quad E_{ml} E_{ij} &= E_{ij} E_{ml} && \text{if } i < m < l < j \text{ or } i < j < m < l, \\ (Y.2) \quad E_{ml} E_{ij} &= q^{-1} E_{ij} E_{ml} && \text{if } m = i \text{ or } l = j, \\ (Y.3) \quad E_{ml} E_{ij} &= E_{ij} E_{ml} + \hat{q} E_{il} E_{mj} && \text{if } i < m < j < l, \\ (Y.4) \quad E_{ml} E_{ij} &= q E_{ij} E_{ml} - \omega E_{il} && \text{if } i < j = m < l. \end{aligned}$$

It is not hard to verify that (Y.2) and (Y.3) imply that, for  $i < m < j < l$ ,

$$(3) \quad E_{ml}E_{ij}^{r-1} = E_{ij}^{r-1}E_{ml} + \hat{q}q^2E_{ij}^{r-2}E_{mj}E_{il},$$

while (Y.1), (Y.2) and (Y.4) imply that, for  $i < j = m < l$ ,

$$(4) \quad E_{ml}E_{ij}^{r-1} = E_{ij}^{r-1}E_{ml} + \beta E_{ij}^{r-2}E_{il}$$

for some  $\beta \in k$ . Now the proof of the first equation in the lemma follows by induction on  $(s, t) < (m, l)$ . The proof of the second one follows similarly. Let  $(s, t) = (1, 2)$ ; then if  $m = 1$  or  $l = 2$  the result follows from (Y.1) and (Y.2). If  $2 < m$ , then by (3)

$$\begin{aligned} E_{ml}E_{12}^{r-1} & \prod_{(1,2) < (i,j)} E_{ij}^{r-1} \\ &= E_{12}^{r-1}E_{ml} \prod_{(1,2) < (i,j)} E_{ij}^{r-1} + \hat{q}q^2E_{12}^{r-2}E_{m2}E_{1l} \prod_{(1,2) < (i,j)} E_{ij}^{r-1}. \end{aligned}$$

Now apply (Y.2) to commute  $E_{1l}$  with  $E_{1j}$  in the second summand, and then use the fact that  $E_{1l}^{r-1}$  appears in the product to conclude that this summand equals zero. Similarly, if  $m = 2$  then use (4). Thus the result follows for  $(1, 2)$ . Assume the lemma holds for all  $(i, j) < (s, t)$ . Then, by the induction assumption,

$$E_{ml} \prod_{(i,j) < (m,l)} E_{ij}^{r-1} = \prod_{(i,j) < (s,t)} E_{ij}^{r-1}E_{ml}E_{st}^{r-1} \prod_{(s,t) < (i,j)} E_{ij}^{r-1}.$$

Now, if  $s < m < l < t$  or  $s < t < m < l$  or  $l = t$  or  $s = m$ , then the result follows easily from (Y.2). If  $s < m < t < l$ , then by (3)

$$\begin{aligned} & \prod_{(i,j) < (s,t)} E_{ij}^{r-1}E_{ml}E_{st}^{r-1} \prod_{(s,t) < (i,j)} E_{ij}^{r-1} \\ &= \prod_{(i,j) < (s,t)} E_{ij}^{r-1}E_{st}^{r-1}E_{ml} \prod_{(s,t) < (i,j)} E_{ij}^{r-1} + \hat{q}q^2E_{st}^{r-2}E_{mt}E_{sl} \prod_{(s,t) < (i,j)} E_{ij}^{r-1}. \end{aligned}$$

But  $(s, t) < (s, l)$ , and hence  $E_{sl}^{r-1}$  is a factor in the second summand. Now apply (Y.2) to commute  $E_{sl}$  with  $E_{sj}$  for  $s < j < l$  and show that the second summand equals zero. Similarly, if  $m = t$ , then use (4).  $\square$

**Proposition 2.1.2.** *Set  $\mathbf{I} := \{0, 1, \dots, r-1\}^T$  and order it lexicographically reading left to right. Denote  $(m(i, j))_{(i,j) \in T} \in \mathbf{I}$  by  $\mathbf{m}$ . Let*

$$0 \neq a := \sum_{\mathbf{m}_0 \leq \mathbf{m}} c_{\mathbf{m}} \prod_{(i,j)} E_{ij}^{m(i,j)},$$

where  $c_{\mathbf{m}} \in kG(U)$  for all  $\mathbf{m}$  and  $c_{\mathbf{m}_0} \neq 0$ . Suppose that  $E_{ml}a = 0$  for all  $1 \leq m < l \leq n-1$ . Then,  $m_0(i, j) = r-1$  for all  $(i, j) \in T$  i.e.  $a = c \prod_{(i,j)} E_{ij}^{r-1}$  for some  $c \in kG(U) - \{0\}$ .

*Proof.* First note that by (1) for any  $(m, l) \in T$  and  $c \in kG(U)$  there exists  $c' \in kG(U)$  such that  $cE_{ml} = E_{ml}c'$ . The proof of the proposition follows now by induction on the set  $T$  ordered lexicographically reading left to right. Since

$$0 = E_{12}a = \sum_{\mathbf{m}_0 \leq \mathbf{m}} c'_{\mathbf{m}} E_{12}^{m(1,2)+1} \prod_{(1,2) < (i,j)} E_{ij}^{m(i,j)}$$



it follows by Theorem 1.1 that  $m_0(1, 2) = r - 1$ . Assume  $m_0(i, j) = r - 1$  for all  $(i, j) < (m, l)$ ; that is,

$$a = \prod_{(i,j) < (m,l)} E_{ij}^{r-1} \sum_{\mathbf{m}_0 \leq \mathbf{m}} c_{\mathbf{m}} \prod_{(m,l) \leq (i,j)} E_{ij}^{m(i,j)}.$$

Now, by Lemma 2.1.1,

$$0 = E_{ml}a = \prod_{(i,j) < (m,l)} E_{ij}^{r-1} \sum_{\mathbf{m}_0 \leq \mathbf{m}} c''_{\mathbf{m}} E_{ml}^{m(m,l)+1} \prod_{(m,l) < (i,j)} E_{ij}^{m(i,j)}.$$

Therefore it follows by Theorem 1.1 that  $m_0(m, l) = r - 1$  as well. This concludes the proof of the proposition.  $\square$

**Corollary 2.1.3.** *The elements*

$$t := \left( \sum_{g \in G(U)} g \right) \prod_{(i,j)} E_{ij}^{r-1} \quad \text{and} \quad t' := \prod_{(i,j)} E_{ij}^{r-1} \left( \sum_{g \in G(U)} g \right)$$

are non-zero left and right integrals of  $U^+$  respectively.

*Proof.* It is enough to show that  $K_i t = t$  for all  $1 \leq i \leq n - 1$  and that  $E_{ij} t = 0$  for all  $(i, j) \in T$ . The first equation follows since  $\sum_{g \in G(U)} g$  is known to be a two-sided integral of  $kG(U)$ , while the second equation follows from Lemma 2.1.1 and (1). Similarly,  $t'$  is a right integral of  $U^+$ .  $\square$

**Proposition 2.1.4.** *Let  $0 \neq b = \sum_{\mathbf{m}_0 \leq \mathbf{m}} d_{\mathbf{m}} \prod_{(i,j)} F_{ij}^{m(i,j)}$ , where  $d_{\mathbf{m}} \in U^+$  for all  $\mathbf{m} \in \mathbf{I}$  and  $d_{\mathbf{m}_0} \neq 0$ . Suppose that  $b \bar{F}_{m,l} = 0$  for all  $1 \leq m < l \leq n - 1$ . Then  $m_0(i, j) = r - 1$  for all  $(i, j) \in T$  i.e.  $b = d \prod_{(i,j)} E_{ij}^{r-1}$  for some  $0 \neq d \in U^+$ .*

*Proof.* The proof is similar to the proof of Proposition 2.1.2.  $\square$

We are ready now to state and prove the main result of this subsection.

**Theorem 2.1.5.** *The element*

$$\Lambda := \left( \sum_{g \in G(U)} g \right) \prod_{(i,j)} E_{ij}^{r-1} \prod_{(i,j)} F_{ij}^{r-1}$$

is a non-zero two-sided integral of  $U_q(sl_n)'$ . Thus  $U_q(sl_n)'$  is unimodular for all  $n$  and  $q$ .

*Proof.* Since  $U = U_q(sl_n)'$  is finite dimensional, it has a non-zero left integral  $\Lambda$ . By [NZ],  $U$  is free over  $U^+$ ; hence  $\Lambda = tu$ , where  $t$  is as in Corollary 2.1.3 and  $u \in U$ . Expressing  $u$  as a linear combination of the basis elements as cited in Theorem 1.1 and using the fact that  $t$  spans a one dimensional ideal of  $U^+$ , we get

$$\Lambda = t \sum_{m_0 \leq m} \leq \beta_{\mathbf{m}_0} \prod_{(i,j)} F_{ij}^{m(i,j)},$$

where  $\beta_{\mathbf{m}_0}$  is a non-zero scalar. Let  $\alpha$  be the distinguished grouplike element of  $O_q(SL_n)'$ . Since  $F_{ij}^r = 0$  and  $\alpha$  is multiplicative, we have  $\alpha(F_{ij}) = 0$ ; hence  $\Lambda F_{ij} = \alpha(F_{ij})\Lambda = 0$ . Thus Proposition 2.1.4 implies that  $\Lambda$  has the desired form. Now, by (1),  $K_i$  commutes with  $\prod_{(i,j)} E_{ij}^{r-1} \prod_{(i,j)} F_{ij}^{r-1}$ , and since  $\sum_{g \in G(U)} g$  is an integral of  $kG(U)$  we conclude that  $\Lambda K_i = \Lambda$  for all  $i$ . Moreover,  $\Lambda E_{ij} = \alpha(E_{ij})\Lambda = 0 = \varepsilon(E_{ij})\Lambda$ . Hence  $\Lambda$  is a right integral as well.  $\square$

**2.2. The integrals of  $O_q(SL_n)'$ .** In what follows we set  $J := \{(m, l) | 1 \leq m \neq l \leq n\}$  and order it lexicographically reading left to right.

**Lemma 2.2.1.** *The following holds for all  $(m, l) \in J$ :*

$$\left( \prod_{(i,j) \in J} x_{ij}^{r-1} \right) x_{ml} = x_{ml} \left( \prod_{(i,j) \in J} x_{ij}^{r-1} \right) = 0.$$

*Proof.* We show by induction that, for all  $(m, l) \in J$  and all  $(s, t) \leq (m, l)$ ,

$$x_{ml} \left( \prod_{(i,j) < (m,l)} x_{ij}^{r-1} \right) = \alpha \left( \prod_{(i,j) \leq (s,t)} x_{ij}^{r-1} \right) x_{ml} \left( \prod_{(s,t) < (i,j) < (m,l)} x_{ij}^{r-1} \right)$$

and

$$\left( \prod_{(m,l) < (i,j)} x_{ij}^{r-1} \right) x_{ml} = \beta \left( \prod_{(m,l) < (i,j) < (s,t)} x_{ij}^{r-1} \right) x_{ml} \left( \prod_{(s,t) \leq (i,j)} x_{ij}^{r-1} \right)$$

for some scalars  $\alpha, \beta \in k$ . It is not hard to verify that (O.3)-(O.6) imply that for  $i < m$  and  $j < l$

$$(5) \quad x_{ml} x_{ij}^{r-1} = x_{ij}^{r-1} x_{ml} + \hat{q} q^{-2} x_{ij}^{r-2} x_{mj} x_{il}.$$

The proof follows now as the proof of Lemma 2.1.1, where (5) stands for both (3) and (4). Take in particular  $(s, t) = (m, l)$  to get  $x_{ml}^r$ , which equals zero. The second part follows similarly.  $\square$

**Lemma 2.2.2.** *Let  $m_1, \dots, m_n \in \{0, 1, \dots, N-1\}$ . Then*

1.

$$\left( \prod_{(i,j) \in J} x_{ij}^{r-1} \right) \left( \prod_{i=1}^n x_{ii}^{m_i} \right) x_{ml} = 0$$

for all  $(m, l) \in J$ , and

2.

$$\left( \prod_{(i,j) \in J} x_{ij}^{r-1} \right) \left( \prod_{i=1}^n x_{ii}^{m_i} \right) x_{jj} = \left( \prod_{(i,j) \in J} x_{ij}^{r-1} \right) \left( \prod_{i=1}^{j-1} x_{ii}^{m_i} x_{jj}^{m_j+1} \prod_{i=j+1}^n x_{ii}^{m_i} \right)$$

for all  $1 \leq j \leq n$ .

*Proof.* 1. Set  $a := \prod_{(i,j) \in J} x_{ij}^{r-1}$ . We present the set  $J$  as the union  $J = J^+ \cup J^-$ , where  $J^+ := \{(i, j) | i < j\}$  and  $J^- := \{(i, j) | i > j\}$ . Order  $J^+$  by  $(i, j) < (k, l)$  if and only if  $i < k$  or  $(i = k \text{ and } l < j)$ , and order  $J^-$  by  $(i, j) < (k, l)$  if and only if  $k < i$  or  $(k = i \text{ and } j < l)$ . We first use induction on  $J^+$ . By (O.3)-(O.5)

$$a \prod_{i=1}^n x_{ii}^{m_i} x_{1n} = q^{m_1 - m_n} a x_{1n} \prod_{i=1}^n x_{ii}^{m_i},$$

and by Lemma 2.2.1 we are done. Assume the lemma holds for all  $(i, j) < (s, t)$ . We claim that

$$a \prod_{i=1}^n x_{ii}^{m_i} x_{st} = q^{m_s - m_t} a x_{st} \prod_{i=1}^n x_{ii}^{m_i}.$$

Indeed, we shift  $x_{st}$  to the left, where for  $i = s$  and  $i = t$  we use (O.3) and (O.4), respectively, and for  $s < i < t$  we use (O.5). Otherwise, for  $s < t < i$  or  $i < s$  we have by (O.6)  $x_{st}x_{ii} = x_{ii}x_{st} + \hat{q}x_{it}x_{si}$ . But if  $s < t < i$  then  $(s, i) \in J^+$  and  $(s, i) < (s, t)$ , and if  $i < s$  then  $(i, s) \in J^+$  and  $(i, s) < (s, t)$ . In any event, by our assumption the second summand vanishes, which proves our claim. The result now follows by Lemma 2.2.1. The induction on  $J^-$  follows similarly.

2. By (O.6),  $x_{ii}x_{jj} = x_{jj}x_{ii} - \hat{q}x_{ij}x_{ji}$  for all  $1 \leq i < j < n$ . Using the first part, we get the desired result.  $\square$

We are ready now to prove the following:

**Theorem 2.2.3.** *Set  $\mathbf{N} := \{0, 1, \dots, N-1\}^{n-1}$ . Then the element*

$$\lambda := \left( \prod_{(i,j) \in J} x_{ij}^{r-1} \right) \left( \sum_{\mathbf{m} \in \mathbf{N}} \prod_{i=1}^n x_{ii}^{m_i} \right)$$

*is a non-zero right integral for  $O_q(SL_n)'$ .*

*Proof.* By Lemma 2.2.2(1) it is enough to show that  $\lambda x_{tt} = \lambda$  for all  $t$ . But by Lemma 2.2.2(2)

$$\lambda x_{tt} = a \sum_{\mathbf{m} \in \mathbf{N}} \prod_{i < t} x_{ii}^{m_i} x_{tt}^{m_t+1} \prod_{i > t} x_{ii}^{m_i} = \lambda,$$

and the result follows.  $\square$

Similarly one can prove the following:

**Theorem 2.2.4.** *The element*

$$\lambda := \left( \sum_{\mathbf{m} \in \mathbf{N}} \prod_{i=1}^n x_{ii}^{m_i} \right) \left( \prod_{(i,j) \in J} x_{ij}^{r-1} \right)$$

*is a non-zero left integral for  $O_q(SL_n)'$ .*

Knowing the left and right integrals of  $O_q(SL_n)'$  enables us to determine the distinguished grouplike element of  $U_q(sl_n)'$ .

**Proposition 2.2.5.** *The element*

$$g := K_1^{4(n-1)} \left( \prod_{i=2}^{n-2} K_i^{8(n-2)} \right) K_{n-1}^{4(n-1)}$$

*is the distinguished grouplike element of  $U_q(sl_n)'$ .*

*Proof.* We wish to compute  $x_{ss}\lambda$  for  $1 \leq s \leq n$ . Now, if  $i < s < l$  then  $x_{ss}$  and  $x_{ij}^{r-1}$  commute, and if  $i = s$  and  $j = s$  then we use (O.3) and (O.4) respectively. Otherwise, if  $i < s$  and  $j < s$  then (O.5) and Lemma 2.2.1 imply that  $x_{ss}$  and  $x_{ij}^{r-1}$  of the product commute. Combining these relations yields

$$x_{ss} \left( \prod_{(i,j) \in J} x_{ij}^{r-1} \right) \left( \sum_{\mathbf{m} \in \mathbf{N}} \prod_{i=1}^n x_{ii}^{m_i} \right) = q^{-2(2s-n-1)} \left( \prod_{(i,j) \in J} x_{ij}^{r-1} \right) \left( \sum_{\mathbf{m} \in \mathbf{N}} \prod_{i=1}^n x_{ii}^{m_i} \right).$$

But, by (2),  $\langle x_{ss}, K_i \rangle = \omega^{\delta_{s,i} - \delta_{s,i+1}}$  for all  $i$  and  $s$ , thus,  $\langle x_{ss}, g \rangle = q^{-2(2s-n-1)}$ , and the result follows.  $\square$

As a result we have the following theorem:

**Theorem 2.2.6.** 1.  $U_q(sl_n)'$  is a ribbon Hopf algebra if and only if  $(n, N) = 1$  or 2.  
 2. The element  $G := K_1^{2(n-1)} \left( \prod_{i=2}^{n-2} K_i^{4(n-2)} \right) K_{n-1}^{2(n-1)}$  is the special grouplike element of  $U_q(sl_n)'$ .

*Proof.* By Proposition 2.2.5 we have  $l^2 = g$ , and it is not hard to verify that  $s^2(a) = lal^{-1}$  for all  $a \in U_q(sl_n)'$ . Since  $U_q(sl_n)'$  is always unimodular by Theorem 2.1.5 and it is quasitriangular if and only if  $(n, N) = 1$  or 2 [GW, Theorem 3.7], the result follows by [KR1, Proposition 3].  $\square$

**2.3. When is  $U_q(sl_n)'$  factorizable?** After establishing a necessary and sufficient condition for  $U_q(sl_n)'$  to be ribbon and unimodular, we turn to the question of its factorizability.

**Lemma 2.3.1.** Let  $f : A^{*cop} \rightarrow A$  be a Hopf algebra map. Then the image of the map  $F := f^* * f$  is an  $(A^*, \leftarrow)$ -sub-module of  $A$ .

*Proof.* Let  $p, q \in A^*$ . Since,

$$\begin{aligned} F(q) \leftarrow p &= p(F(q)_{(1)})F(q)_{(2)} \\ &= p([f^*(q_{(1)})f(q_{(2)})]_{(1)})[f^*(q_{(1)})f(q_{(2)})]_{(2)} \\ &= p(f^*(q_{(1)})_{(1)}f(q_{(2)})_{(1)})f^*(q_{(1)})_{(2)}f(q_{(2)})_{(2)} \\ &= p(f^*(q_{(1)})f(q_{(4)}))f^*(q_{(2)})f(q_{(3)}) \\ &= p(f^*(q_{(1)})f(q_{(3)}))F(q_{(2)}), \end{aligned}$$

the result follows.  $\square$

In the following we prove a useful test for factorizability.

**Theorem 2.3.2.** Let  $(A, R_f)$  be a finite dimensional quasitriangular Hopf algebra, and set  $F := f^* * f$ . Let  $\Lambda, \lambda$  be non-zero right integrals of  $A$  and  $A^*$  respectively. Then  $A$  is factorizable if and only if  $F(\lambda) = \alpha\Lambda$  for some non-zero scalar  $\alpha \in k$ .

*Proof.* Suppose  $A$  is factorizable; that is,  $F$  is an isomorphism. Then  $\Lambda$  is a non-zero two-sided integral [R1, Proposition 3]. Let  $p \in A^*$  be so that  $F(p) = \Lambda$ . We wish to show that  $p$  is a non-zero right integral of  $A^*$ . Indeed, using the fact that  $f$  is an anti-coalgebra and algebra map and that  $f^*$  is an anti-algebra and coalgebra map, we derive

$$\begin{aligned} F(pq) &= f^*(p_{(1)}q_{(1)})f(p_{(2)}q_{(2)}) \\ &= f^*(q_{(1)})f^*(p_{(1)})f(p_{(2)})f(q_{(2)}) = f^*(q_{(1)})F(p)f(q_{(2)}) \\ &= \varepsilon(f^*(q_{(1)}))F(p)f(q_{(2)}) = F(p)\varepsilon(q_{(1)})f(q_{(2)}) \\ &= F(p)f(q) = \varepsilon(f(q))F(p) = F(\varepsilon(q)p) \end{aligned}$$

for any  $q \in A^*$ . Since  $F$  is one-to-one, the result follows.

Conversely, suppose  $\Lambda$  belongs to the image of  $F$ . It is well known that  $\Lambda \leftarrow A^* = A$ . Therefore the result follows from Lemma 2.3.1. This concludes the proof of the theorem.  $\square$

**Lemma 2.3.3.** *Let  $f, f^*$  be as in Theorem 1.2. Set  $F := f^* * f$ . Then*

$$\begin{aligned} F & \left[ \prod_{(i,j) \in J^+} x_{ij}^{r-1} \prod_{(t,l) \in J^-} x_{tl}^{r-1} \prod_{s=1}^n x_{ss}^{\lambda_s} \right] \\ & = (f^* \otimes f) \left[ \prod_{(i,j) \in J^+} (x_{ij}^{r-1} \otimes x_{jj}^{r-1}) \prod_{(t,l) \in J^-} (x_{tt}^{r-1} \otimes x_{tl}^{r-1}) \prod_{s=1}^n (x_{ss}^{\lambda_s} \otimes x_{ss}^{\lambda_s}) \right]. \end{aligned}$$

*Proof.* We compute:

$$\begin{aligned} \Delta & \left[ \prod_{(i,j) \in J^+} x_{ij}^{r-1} \prod_{(t,l) \in J^-} x_{tl}^{r-1} \prod_{s=1}^n x_{ss}^{\lambda_s} \right] \\ & = \prod_{(i,j) \in J^+} \Delta(x_{ij})^{r-1} \prod_{(t,l) \in J^-} \Delta(x_{tl})^{r-1} \prod_{s=1}^n \Delta(x_{ss})^{\lambda_s} \\ & = \prod_{(i,j) \in J^+} \left( \sum_{a=1}^n x_{ia} \otimes x_{aj} \right)^{r-1} \prod_{(t,l) \in J^-} \left( \sum_{b=1}^n x_{tb} \otimes x_{bl} \right)^{r-1} \prod_{s=1}^n \left( \sum_{c=1}^n x_{sc} \otimes x_{cs} \right)^{\lambda_s} \\ & = \prod_{(i,j) \in J^+} \left( \sum_{\bar{a} \in \{1, \dots, n\}^{r-1}} x_{i\bar{a}(1)} \cdots x_{i\bar{a}(r-1)} \otimes x_{\bar{a}(1)j} \cdots x_{\bar{a}(r-1)j} \right) \\ & \quad \times \prod_{(t,l) \in J^-} \left( \sum_{\bar{b} \in \{1, \dots, n\}^{r-1}} x_{t\bar{b}(1)} \cdots x_{t\bar{b}(r-1)} \otimes x_{\bar{b}(1)l} \cdots x_{\bar{b}(r-1)l} \right) \\ & \quad \times \prod_{s=1}^n \left( \sum_{\bar{c} \in \{1, \dots, n\}^{\lambda_s}} x_{s\bar{c}(1)} \cdots x_{s\bar{c}(r-1)} \otimes x_{\bar{c}(1)s} \cdots x_{\bar{c}(r-1)s} \right) \\ & = \sum_{\bar{a} \in \{1, \dots, n\}^{(r-1) \times J^+}} \prod_{(i,j) \in J^+} (x_{i\bar{a}(1)} \cdots x_{i\bar{a}(r-1)} \otimes x_{\bar{a}(1)j} \cdots x_{\bar{a}(r-1)j}) \\ & \quad \times \sum_{\bar{b} \in \{1, \dots, n\}^{(r-1) \times J^-}} \prod_{(t,l) \in J^-} (x_{t\bar{b}(1)} \cdots x_{t\bar{b}(r-1)} \otimes x_{\bar{b}(1)l} \cdots x_{\bar{b}(r-1)l}) \\ & \quad \times \sum_{\bar{c} \in \{1, \dots, n\}^{(\lambda_s) \times n}} \prod_{s=1}^n (x_{s\bar{c}(1)} \cdots x_{s\bar{c}(\lambda_s)} \otimes x_{\bar{c}(1)s} \cdots x_{\bar{c}(\lambda_s)s}) \\ & = \sum_{\bar{a}, \bar{b}, \bar{c}} \prod_{(i,j) \in J^+} \prod_{(t,l) \in J^-} \prod_{s=1}^n x_{i\bar{a}(1)} \cdots x_{i\bar{a}(r-1)} x_{t\bar{b}(1)} \cdots x_{t\bar{b}(r-1)} x_{s\bar{c}(1)} \\ & \quad \cdots x_{s\bar{c}(\lambda_s)} \otimes x_{\bar{a}(1)j} \cdots x_{\bar{a}(r-1)j} x_{\bar{b}(1)l} \cdots x_{\bar{b}(r-1)l} x_{\bar{c}(1)s} \cdots x_{\bar{c}(\lambda_s)s}. \end{aligned}$$

Next, we prove by induction that

$$(f^* \otimes f) \left( \prod_{(i,j) \in J^+} \prod_{(t,l) \in J^-} \prod_{s=1}^n x_{i\bar{a}_{ij}(1)} \cdots x_{i\bar{a}_{ij}(r-1)} x_{t\bar{b}_{tl}(1)} \cdots x_{t\bar{b}_{tl}(r-1)} x_{s\bar{c}_s(\lambda)_s} \right. \\ \left. \cdots x_{\bar{c}_s(\lambda)_s} \otimes x_{\bar{a}_{ij}(1)j} \cdots x_{\bar{a}_{ij}(r-1)j} x_{\bar{b}_{tl}(1)l} \cdots x_{\bar{b}_{tl}(r-1)l} x_{\bar{c}_s(1)s} \cdots x_{\bar{c}_s(\lambda_s)s} \right) \neq 0$$

if and only if  $\bar{a}_{ij}(d) = j$ ,  $\bar{b}_{tl}(d) = t$  and  $\bar{c}_s(e) = s$  for all  $1 \leq d \leq r-1$  and  $1 \leq e \leq \lambda_s$ . We shall prove that  $\bar{a}_{ij}(d) = j$  for all  $1 \leq d \leq r-1$ . The rest can be proved similarly. For  $(i, j) = (2, 1)$  we have  $f(x_{\bar{a}_{21}(d)1}) \neq 0$  if and only if  $\bar{a}_{21}(d) = 1$  for all  $d$ . Assume that for all  $(u, v) < (i, j)$  we have proved that  $\bar{a}_{uv}(d) = v$  for all  $d$ . That is,  $\prod_{(u,v) < (i,j)} x_{uv}^{r-1}$  is a factor of the product. Now, for  $(i, j)$ ,  $f(x_{i\bar{a}_{ij}(d)}) \neq 0$  implies that  $\bar{a}_{ij}(d) \leq j$ . But if  $\bar{a}_{ij}(d) < j$  then  $(i, \bar{a}_{ij}(d)) < (i, j)$ , and by assumption  $x_{i\bar{a}_{ij}(d)}^{r-1}$  is a factor in the product. Hence by Lemma 2.2.1 these summands vanish, and we are done.  $\square$

We are ready now to determine when  $U_q(sl_n)'$  is factorizable.

**Theorem 2.3.4.** *Let  $N$  be the order of  $q^{1/2}$ . Then  $U_q(sl_n)'$  is factorizable if and only if  $(n, N) = 1$  and  $N$  is odd; that is, if and only if it is minimal quasitriangular.*

*Proof.* Suppose  $U_q(sl_n)'$  is factorizable. Since it is quasitriangular, it follows by Theorem 1.2 that either  $(n, N) = 1$  or  $(n, N) = 2$ . Since by [R1, Proposition 3] factorizable implies minimal quasitriangular, the result follows from Theorem 1.2. Conversely, suppose  $(n, N) = 1$  and  $N$  is odd. Then by Theorem 1.2  $U_q(sl_n)'$  admits only two non-isomorphic minimal quasitriangular structures. We prove the theorem for the quasitriangular structure  $(U_q(sl_n)', R_f)$  as given in Theorem 1.2. The proof for the other structure follows similarly. Using Lemma 2.3.3, we compute

$$F \left[ \prod_{i>j} x_{ij}^{r-1} \prod_{t<l} x_{tl}^{r-1} \left( \sum_{\lambda=(\lambda_s) \in \{1, \dots, n\}^{\{0, \dots, N-1\}}} \prod_{s=1}^n x_{ss}^{\lambda_s} \right) \right] \\ = \sum_{\lambda} F \left[ \prod_{i>j} x_{ij}^{r-1} \prod_{t<l} x_{tl}^{r-1} \prod_{s=1}^n x_{ss}^{\lambda_s} \right] \\ = \sum_{\lambda} (f^* \otimes f) \left[ \prod_{i>j} (x_{ij}^{r-1} \otimes x_{jj}^{r-1}) \prod_{t<l} (x_{tt}^{r-1} \otimes x_{tl}^{r-1}) \prod_{s=1}^n (x_{ss}^{\lambda_s} \otimes x_{ss}^{\lambda_s}) \right] \\ = \sum_{\lambda} f^* \left( \prod_{i>j} x_{ij}^{r-1} \prod_{t=1}^n x_{tt}^{(r-1)(n-t)} \prod_{s=1}^n x_{ss}^{\lambda_s} \right) f \left( \prod_{j=1}^n x_{jj}^{(r-1)(n-j)} \prod_{t<l} x_{tl}^{r-1} \prod_{s=1}^n x_{ss}^{\lambda_s} \right) \\ = \sum_{\lambda} f^* \left( \prod_{s=1}^n x_{ss}^{\lambda_s} \right) f^* \left( \prod_{t=1}^n x_{tt}^{(r-1)(n-t)} \right) f^* \left( \prod_{i>j} x_{ij}^{r-1} \right) \\ \times f \left( \prod_{j=1}^n x_{jj}^{(r-1)(n-j)} \right) f \left( \prod_{t<l} x_{tl}^{r-1} \right) f \left( \prod_{s=1}^n x_{ss}^{\lambda_s} \right)$$

$$\begin{aligned}
 &= \alpha \sum_{\lambda} \left( \prod_{s=1}^n g_{ss}^{\lambda_s} \right) \left( \prod_{t=1}^n g_{tt}^{(r-1)(n-t)} \right) \left( \prod_{i>j} g_{ij}^{r-1} E_{ji}^{r-1} \right) \\
 &\quad \times \left( \prod_{j=1}^n g_{jj}^{(r-1)(n-j)} \right) \left( \prod_{t<l} g_{tl}^{r-1} F_{tl}^{r-1} \right) \left( \prod_{s=1}^n g_{ss}^{\lambda_s} \right) \\
 &= \prod_{t=1}^n g_{tt}^{(r-1)(n-t)} \prod_{i>j} g_{ij}^{r-1} E_{ji}^{r-1} \prod_{j=1}^n g_{jj}^{(r-1)(n-j)} \prod_{t<l} g_{tl}^{r-1} F_{tl}^{r-1} \left( \alpha \sum_{\lambda} \prod_{s=1}^n g_{ss}^{2\lambda_s} \right) \\
 &= \alpha \prod_{i>j} E_{ji}^{r-1} \prod_{t<l} F_{tl}^{r-1} \left( \sum_{\lambda} \prod_{s=1}^n g_{ss}^{2\lambda_s} \right) \\
 &= \alpha \left( \sum_{g \in G(U)} g \right) \prod_{(i,j)} E_{ij}^{r-1} \prod_{(i,j)} F_{ij}^{r-1}.
 \end{aligned}$$

Therefore,  $F(\lambda) = \alpha \Lambda$ , and the result follows from Theorem 2.3.2. Note that the last equation follows by our assumption on the oddness of  $N$ . This concludes the proof of the theorem.  $\square$

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